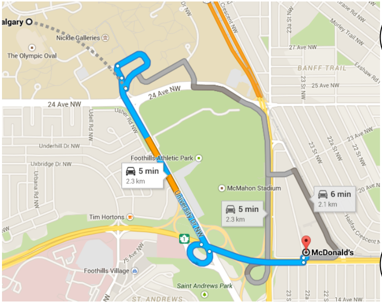
Dijkstra's Algorithm

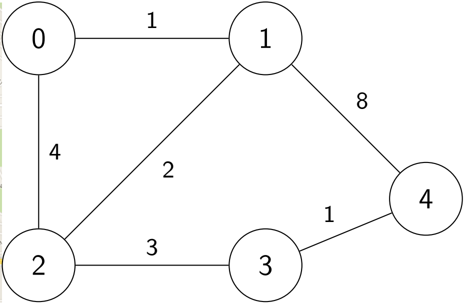
Problem Solving Club

February 1, 2017

Dijkstra's algorithm

- Dijkstra's is a **greedy** algorithm that finds **shortest paths** from a **single source** to **every other vertex** in the graph.
- As usually implemented:
  - Works for **weighted** graphs with **non-negative weights**.
  - Works for **directed** and **undirected** graphs.
  - Runs in $O((V + E) \log V)$.





Binary heap (priority queue) data structure

- A binary heap is a data structure with two operations:
  - **Insert:** Insert an element into the heap.
  - **Extract:** Remove the max (or min) element from the heap.
- Both operations take at worst $O(\log N)$.
- C++ std::priority_queue, Java PriorityQueue

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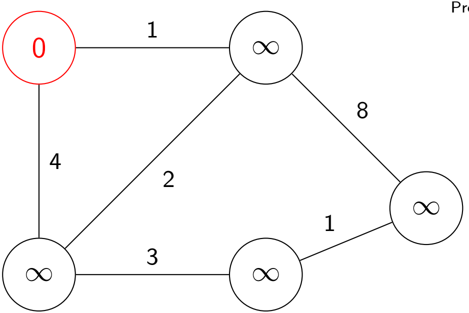
Dijkstra's Algorithm: An Example



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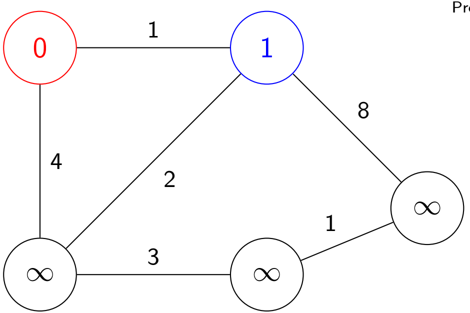

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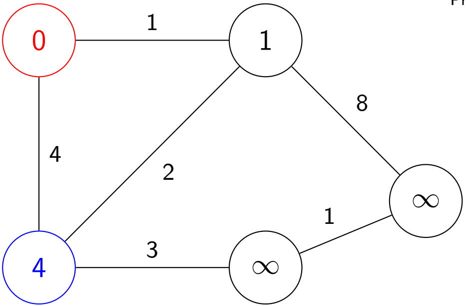

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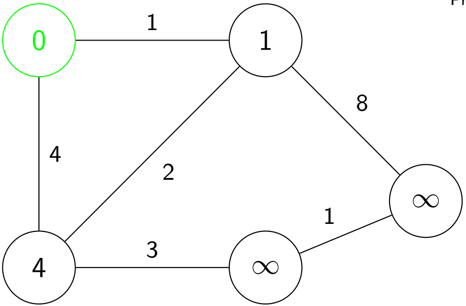

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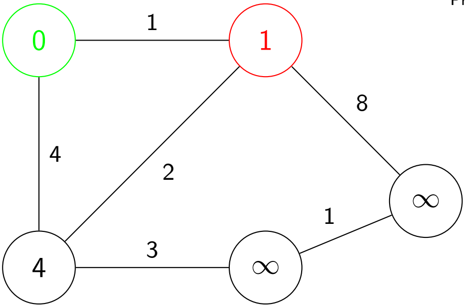

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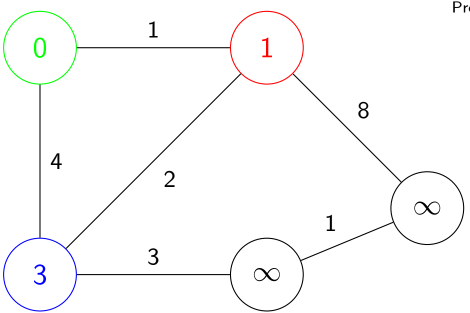

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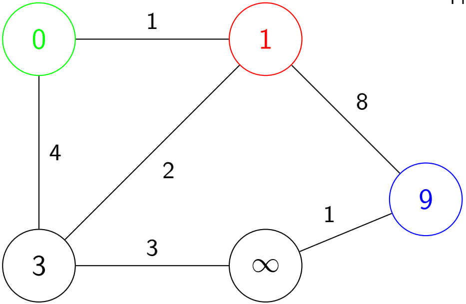

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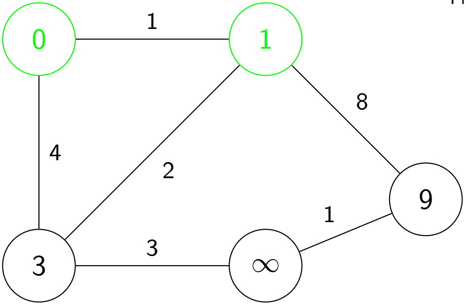

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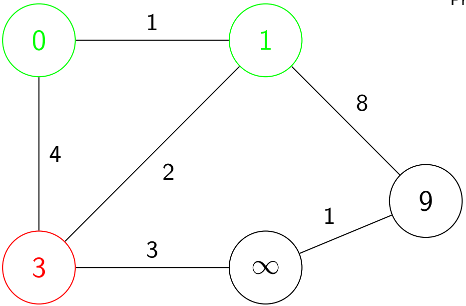

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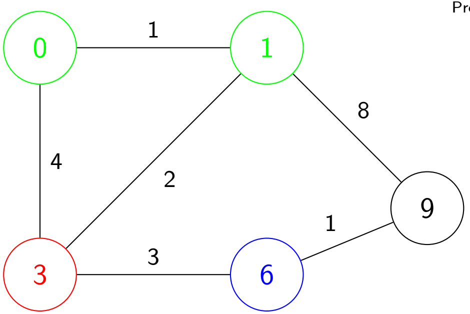

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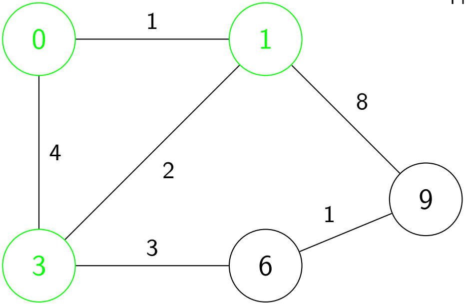

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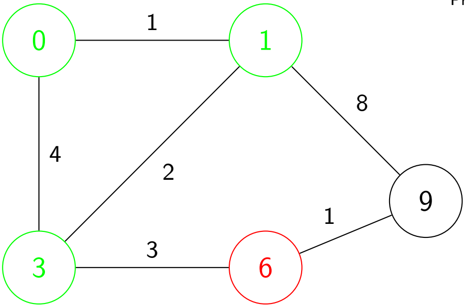

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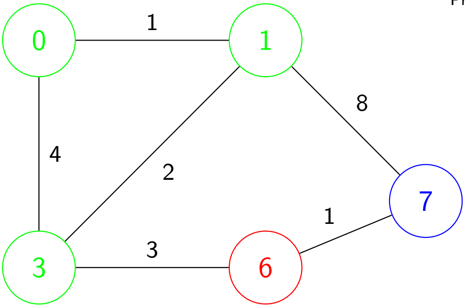

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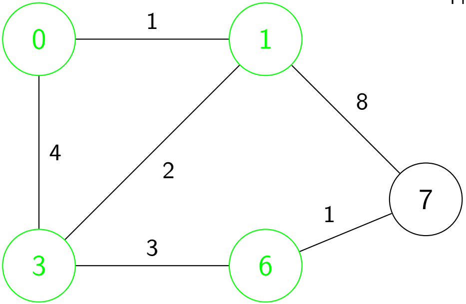

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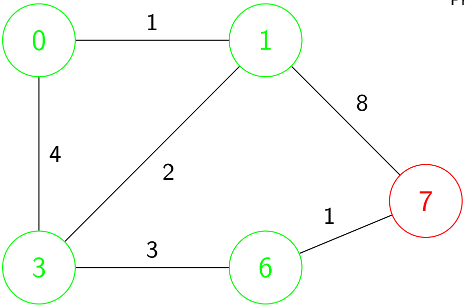

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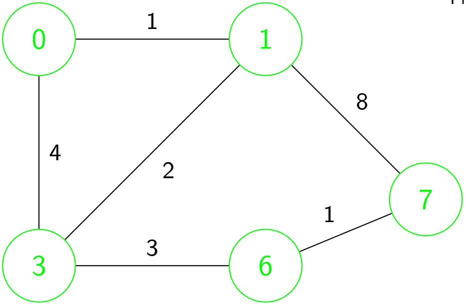

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Example code

```
vector<edge> adj[100];
vector<int> dist(100, INF);

void dijkstra(int start) {
    dist[start] = 0;
    priority_queue<pair<int, int>,
                   vector<pair<int, int> >,
                   greater<pair<int, int> > > pq;
    pq.push(make_pair(dist[start], start));

    while (!pq.empty()) {
        int u = pq.top().second,
            d = pq.top().first;
        pq.pop();
        if (d > dist[u]) continue;
        for (int i = 0; i < adj[u].size(); i++) {
            int v = adj[u][i].v,
                w = adj[u][i].weight;
            if (w + dist[u] < dist[v]) {
                dist[v] = w + dist[u];
                pq.push(make_pair(dist[v], v));
            }
        }
    }
}
```

Frequently asked questions

- How do I find the actual shortest paths?
  - Keep track of each vertex's parent using a separate array.
- Can I use std::set or TreeSet instead of a binary heap?
  - Yes. Same asymptotic performance, but worse in practice.
- Can Dijkstra's find the longest path in a graph?
  - No. Longest path problem for general graph is NP-hard.
- What if my graph has negative weights?
  - Bellman-Ford / Shortest Path Faster Algorithm: $O(VE)$.
    - Same as Dijkstra's, but works with negative weights/cycles.
  - Floyd-Warshall: $O\left(V^3\right)$.
    - Finds the shortest path between every pair of vertices.
  - These have much worse running time than $O((V+E)\log V)$.